TRANSFORMATIONAL GEOMETRY: THE GEOMETRY OF MOTION

MATHINK,
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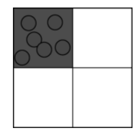
Warm up

A small pizza is 6 inches across and costs $4.99.
The large pizza at the same pizzeria is 12 inches across and costs $11.99.
Which is a better buy?

If it were square, it would be clear that 4 small pizzas is the same amount of food as 1 large pizza.

Is this true for circular pizzas?



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Outline

- What is transformational geometry?
- Transformational geometry in the CaCCSS-M
- Transformations that change only position, not shape or size
- Symmetry
- Similarity and transformations that change size
- Transformational constructions
- Transformations in analytic geometry

What is transformational geometry?

- 3 styles of doing geometry in the plane
  - Synthetic (Euclid, 300 BC)
    - Objects are defined from points, lines. No motion.
    - Proofs use facts about the objects
  - Analytic (from 1650)
    - Objects are defined by coordinates and equations
    - Proofs use algebra
  - Transformational (from 20th century)
    - Emphasis on actions (transformations of the plane): rotation, reflection, translation, scaling, …
    - Proofs use facts about transformations

Transformational geometry in CCSS

Grade 8:

- Understand congruence and similarity using physical models, transparencies, or geometry software.
  - … rotations, reflections, translations, dilations…

High School:

- Experiment with transformations in the plane
- Understand congruence in terms of rigid motions
- Make geometric constructions
- Understand similarity in terms of similarity transformations

Physical Math, with transformations that don't change shape or size

- Act out reflections with a partner
- Act out 180 degree rotations with a partner, several different centers
- Other ways to do reflections and rotations:
  - Mirror or Mira™ (reflections)
  - Fold and cut paper (reflections)
  - Transparencies (reflections are a flip, rotations a turn)

Transformations in software

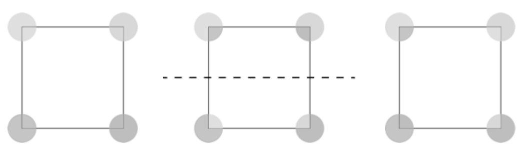
- Dynamic geometry: GeoGebra, Geometer's Sketchpad, Cabri
  - Built-in transformation tools:
- Image and drawing software (Photoshop, CorelDraw, Microsoft Word, etc.)
  - Also have built-in transformations

Symmetry

- A figure or object is symmetric if it fits in its own outline in more than one way.
- Which of these puzzle pieces are symmetric?


Describing symmetries

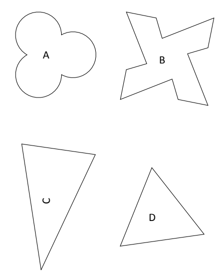
- To describe a symmetry, specify the motion (rotation, reflection, translation…) that moves it from the original position to the new position.



Original position | Reflected across a horizontal line from original position | Rotated 90° from original position

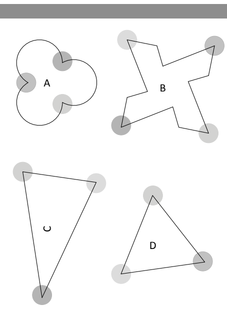
Find all reflection symmetries of these figures.

- [worksheet]
- Tools: Mira/Reflecta, mirror
- Draw the mirror line.



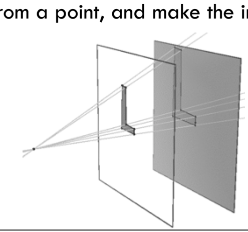
Find all rotation symmetries of these figures.

- [worksheet]
- Color important vertices in different colors.
- Cut out figures, keeping the outline.
- Rotate figure to fit its outline.
- How much did you rotate? (Fraction of complete turn or number of degrees)



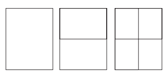
Physical math: similarity and scaling

- Two figures are similar if one is a scaled copy of the other. (Exact same shape, different size.)
- Physical origin: Projector. Light rays spread as they go out from a point, and make the image bigger.



Physical math: Checking for similarity with the Eyeball Test

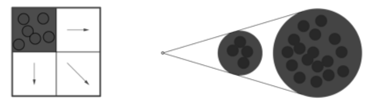
- Close one eye.
- Hold two plane figures in parallel planes, one farther away than the other.
- If the figures line up perfectly, they are similar.
- Try it with 3 rectangles:
  - Full sheet of copier paper, half sheet, quarter sheet



  - Are they all similar? Why or why not?

Scaling a figure

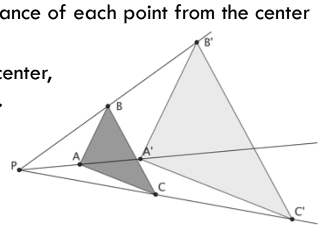
- Scaling means stretching or shrinking uniformly.



- Figures can also be scaled in only one direction.

Scaling a figure

- Scaling in geometry is accomplished by a dilation.
- Choose a scale factor *S* and a center point.
- Multiply the distance of each point from the center point by *S*.
- Shown: P is the center, scale factor is 2.



Did you know?

All parabolas are similar.

- Use the Eyeball Test to check (cut out parabolas)
- The Eyeball Test is not a proof, but here are some ways to prove this:
  - A parabola is the set of all points equidistant from a point and a line. Scale the whole construction
  - Use the inverse transformation principle on $y=x^2$ to get $y=Ax^2$, or vice versa. How is A related to the scale factor?

Examples: constructions with transformations

- What common shape results from these steps? Why?

1. Construct line L, point A not on L.
2. Reflect A across L to make C.
3. Choose B on L but not on AC.
Construct triangle ABC.

Examples: transformational constructions

- What common shape results from these steps? Why?

1. Choose points A and P
2. Rotate A 90° around P to get B.
3. Rotate B 90° around P to get C, and rotate C 90° around P to get D.

Construct quadrilateral ABCD.

Examples: transformational constructions

- What common shape results from these steps? Why?

1. Choose points A, B, C, not collinear
2. Translate C by vector AB to get D.

Construct the quadrilateral ABCD

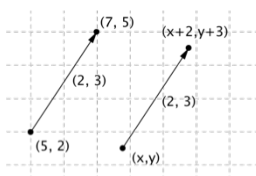
Transformational analytic geometry

- Remainder of presentation: transformations in the coordinate plane
- Questions:
  - When a curve is constructed by transforming a parent graph, how are their equations related?
  - When you graph $y = f(x - h)$ where h is positive, why do you translate the graph of $y = f(x)$ to the *right*?

Translating in the coordinate plane

- How to translate a point by the vector (2,3):

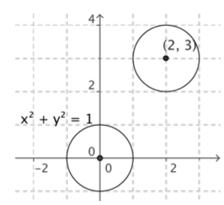
$$(x, y) + (2, 3) = (x + 2, y + 3)$$



Translating a curve

- C = circle with center (2,3) and radius 1
- Assume we know the equation of a circle with center (0,0) and radius 1 (from the Pythagorean Theorem)

$$x^2 + y^2 = 1$$



Translating a curve

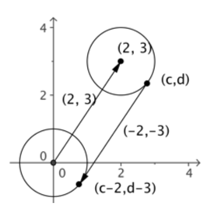
- A point P=(c,d) on the *translated* curve probably doesn't satisfy the original equation $x^2 + y^2 = 1$
- Translate it back to the original curve; now it does.

$$(c, d) - (2, 3) = (c - 2, d - 3)$$

$$(c - 2)^2 + (d - 3)^2 = 1$$

Equation of translated curve is

$$(x - 2)^2 + (y - 3)^2 = 1$$



Translating a curve: try it yourself

Curve with known equation: line through the origin with slope 2 $\quad \frac{y}{x} = 2$

- Find the equation for the line through (-1,3) with slope 2 $\quad \frac{y - 3}{x - (-1)} = 2$

More generally,

- Start with line through the origin with slope m. Find the equation of the line through (h,k) with slope m. $\quad \frac{y}{x} = m \qquad \frac{y - k}{x - h} = m$

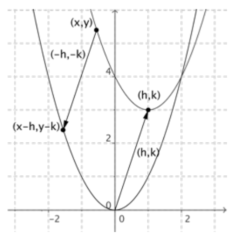
Translating a curve: try it yourself

Curve with known equation: standard parabola with equation $y = x^2$

Find the equation of the parabola with the same shape, size, and orientation, and vertex (h,k)

$$y - k = (x - h)^2$$

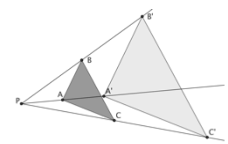
$$y = (x - h)^2 + k$$



Scaling a figure

Scaling in coordinates (with center (0,0)) is accomplished by multiplying both coordinates by the scale factor

$$(x, y) \to (2x, 2x)$$



Scaling a curve

- The same inverse transformation principle applies:
- To find the equation of a scaled curve, scale the coordinates back to a known curve, then substitute.

Scaling a curve

- Find the equation of a circle with center (0,0) and radius 2. (Stretched unit circle, scale factor 2.)
- A general point (x,y) on this curve doesn't satisfy

$$x^2 + y^2 = 1$$

- Shrink it back to the origin by $(x, y) \to (\frac{x}{2}, \frac{y}{2})$
- Substitute into original equation,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

- Equation is equivalent to

$$x^2 + y^2 = 4$$

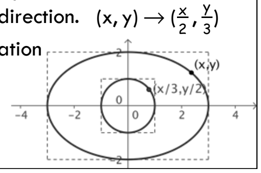
Non-uniform scaling

- Find the equation of an ellipse with center (0,0) and semiaxes of length 2 (horizontal) and 3 (vertical)
- A general point (x,y) on this curve doesn't satisfy

$$x^2 + y^2 = 1$$

- Shrink it back to the origin by a factor of ½ in the x direction and 1/3 in the y direction. $(x, y) \to (\frac{x}{2}, \frac{y}{3})$
- Substitute into original equation

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$



Other transformations

- The same inverse transformation principle holds for any transformation, including
  - Reflection (easy to do for x axis, y axis, line y=x)
  - Rotation (easy to do for 90° and 180°)
- Substitution to simplify equations used to be a main strand of algebra (see textbooks of Leonhard Euler (1707-1783), the progenitor of most of our current curriculum)

Substitution to simplify an equation

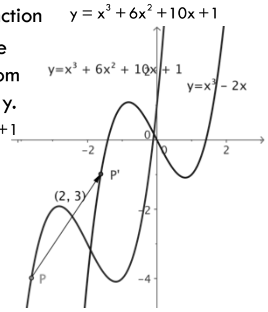
- Graph the polynomial function $y = x^3 + 6x^2 + 10x + 1$
- Translate the graph by the vector (2,3): subtract 2 from all the x's and 3 from the y.

$$y - 3 = (x - 2)^3 + 6(x - 2)^2 + 10(x - 2) + 1$$

- This simplifies to

$$y = x^3 - 2x$$

- Graph, then translate by (-2,-3)



Thank you!

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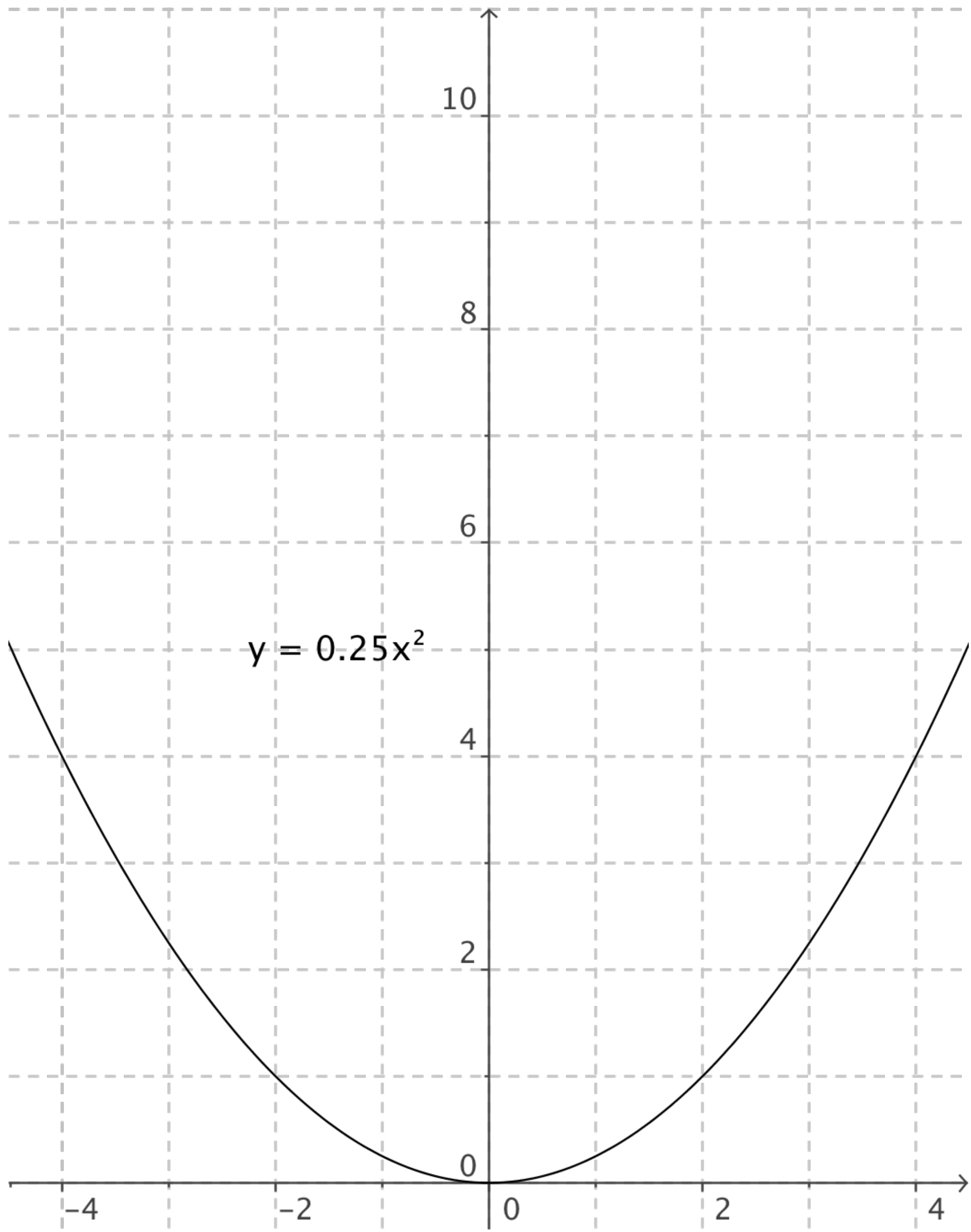
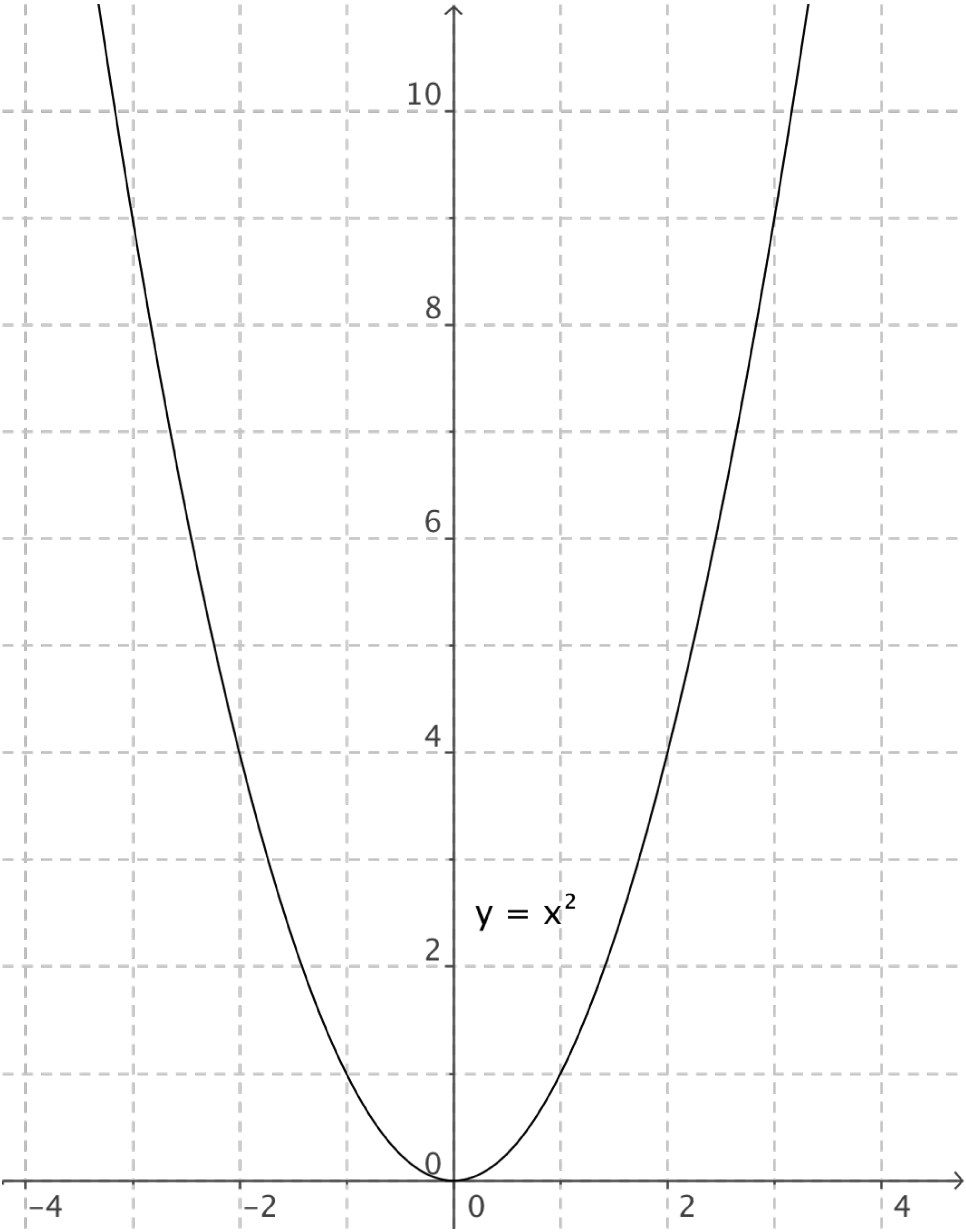
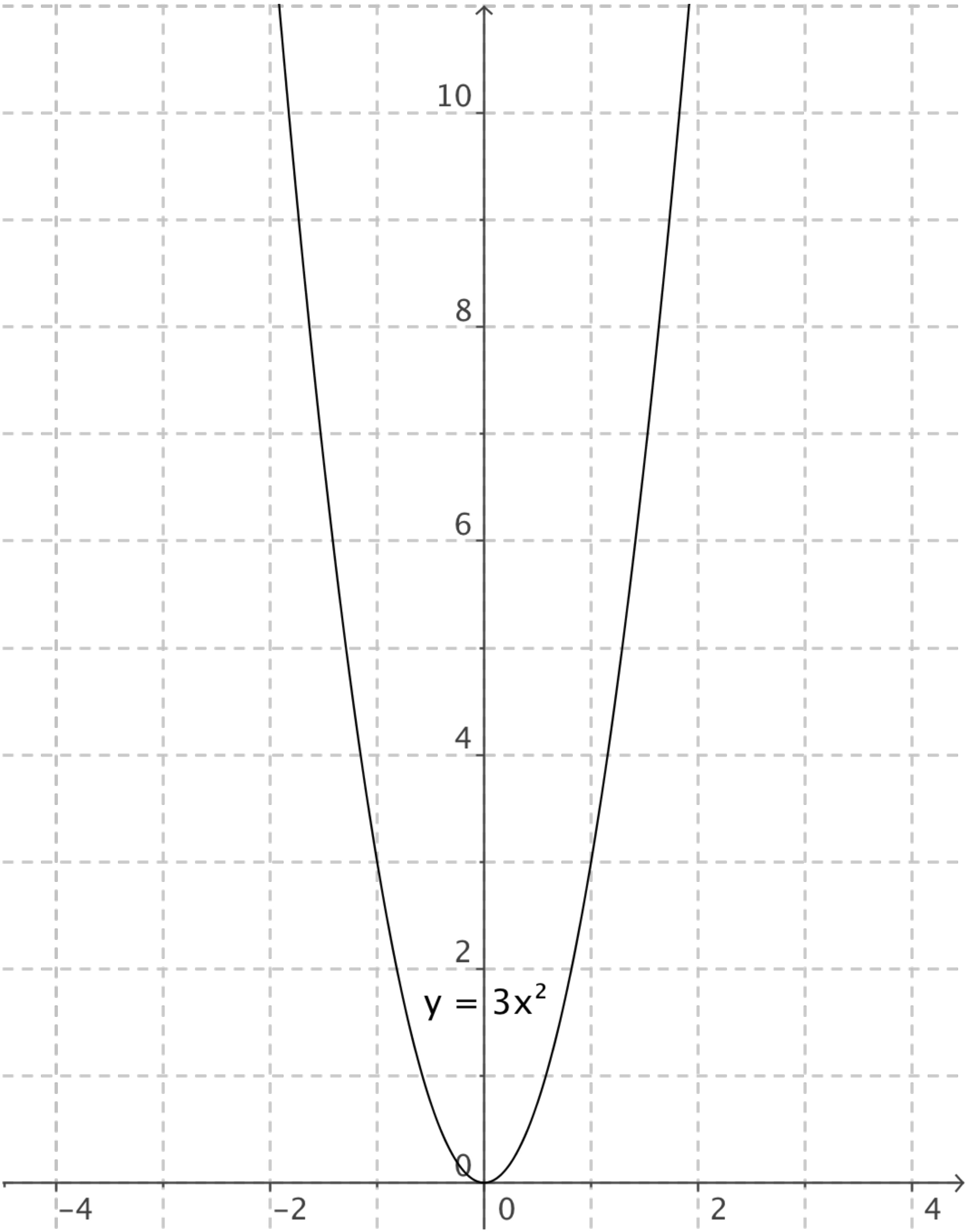


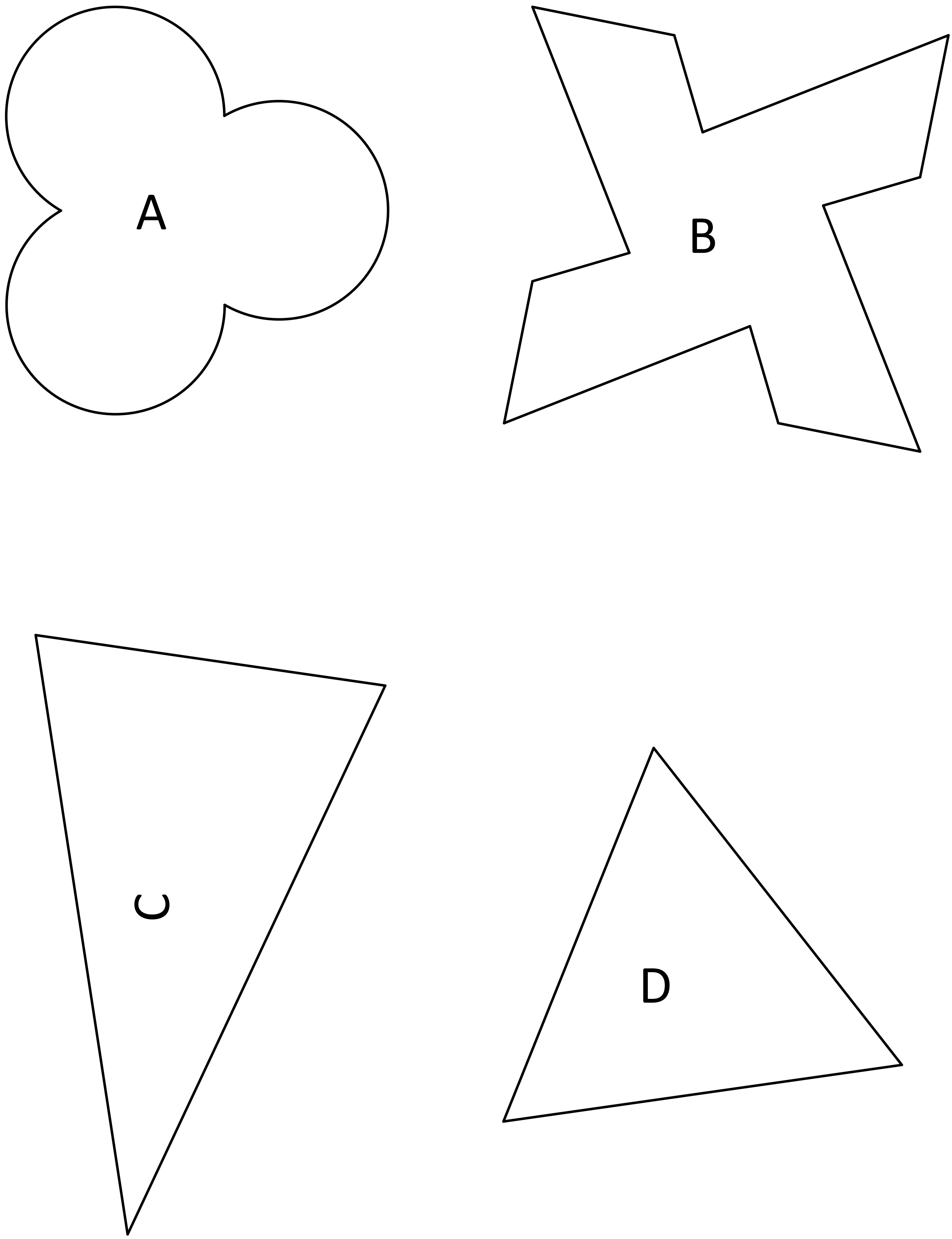
y = 0.25x²
10
8
6
4
2
0
−4
−2
0
2
4

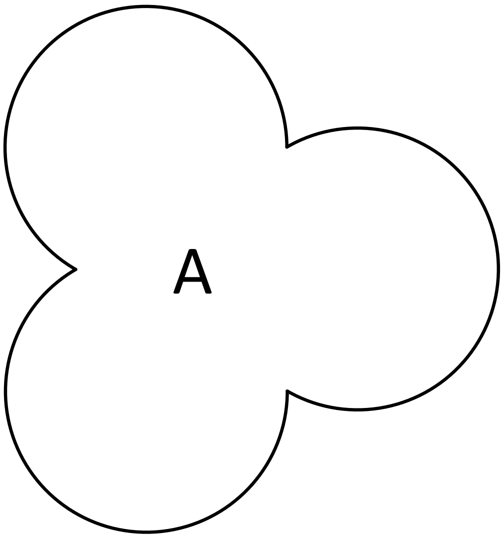


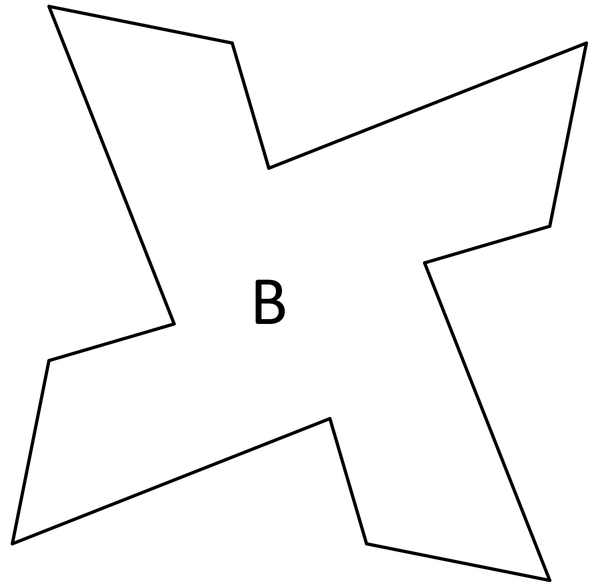
10
8
6
4
2
0
$y = x^2$
−4
−2
0
2
4

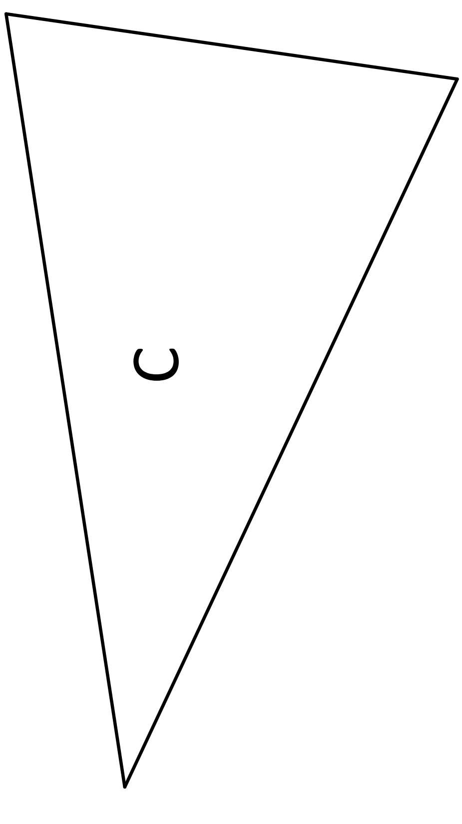


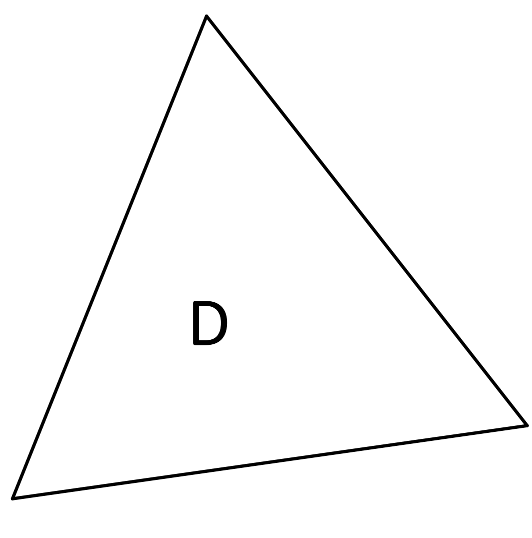
10
8
6
4
2
y = 3x²
0
−4
−2
0
2
4


A
B
C
D


A


B


C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